

# Consistency conditions in the chiral ring of super Yang-Mills theories

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Starting from the generalized Konishi anomaly equations at the non-perturbative level, we demonstrate that the algebraic consistency of the quantum chiral ring of the  $\mathcal{N} = 1$  super Yang-Mills theory with gauge group  $U(N)$ , one adjoint chiral superfield  $X$  and  $N_f \leq 2N$  flavours of quarks implies that the periods of the meromorphic one-form  $\text{Tr} \frac{dz}{z-X}$  must be quantized. This shows in particular that identities in the open string description of the theory, that follow from the fact that gauge invariant observables are expressed in terms of gauge variant building blocks, are mapped onto non-trivial dynamical equations in the closed string description.

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# 1 Introduction

The fact that any four dimensional gauge theory has two seemingly unrelated formulations, one in terms of open strings, which is equivalent to the standard field theoretic Yang-Mills description and the other in terms of closed strings, which thus contains quantum gravity, is an extremely deep and fascinating property. Following [1], many successful examples of this duality have been studied over the last decade. Yet many questions, both technical and conceptual, remain unsolved.

A fundamental conceptual issue is to understand how the basic ingredients in one formulation are encoded in the other formulation and vice-versa. For example, how does the closed string gravity theory know about the Yang-Mills equations of motion? In the closed string description, we do not see the gauge group, for only gauge invariant quantities can be constructed. This is of course not an inconsistency, since the gauge symmetry is really a redundancy in the description of the theory and not a physical symmetry. However, how then can we understand charge quantization à la Dirac, which is usually derived from gauge invariance, in the closed string set-up? A directly related question, which will be at the basis of the present work, is the following. In the open string framework, gauge invariant observables are built in terms of fields that transform non-trivially under the gauge group, and this has some non-trivial mathematical consequences. For example, imagine that the gauge group is  $U(N)$  and that the theory contains an adjoint field  $X$ . The gauge invariant operators built from  $X$  are obtained by considering traces

$$u_k = \text{Tr } X^k \tag{1.1}$$

or product of traces. The fact that  $X$  is a  $N \times N$  matrix implies that there exists homogeneous polynomials  $P_p$  of degree  $N + p$ , if the degree of homogeneity of  $X$  is one, such that

$$u_{N+p} = P_p(u_1, \dots, u_N), \quad p \geq 1. \tag{1.2}$$

Thus only  $u_1, \dots, u_N$  are independent. But how does the closed string theory know about (1.2), while the matrix  $X$  does not exist in the closed string framework? In a sense we are asking how to build the open strings starting from the closed strings, which is a notoriously difficult question.

An extremely interesting incarnation of the open/closed string duality is obtained when one focus on the chiral sector of  $\mathcal{N} = 1$  supersymmetric gauge theories. The closed string set-up involves a geometric transition [2] and is equivalent to the Dijkgraaf-Vafa matrix model description [3]. On the other hand, the model has been solved recently starting from the usual field theoretic description [4, 5, 6], using

Nekrasov's instanton technology [7]. The theory is essentially reduced to a statistical model of colored partitions which, remarkably, yields gauge theory correlators that coincide with the matrix model predictions [5, 6]. The open/closed string duality is thus fully understood in this case. Our aim in the present paper, which is a continuation of [8], is to address some of the above conceptual questions in this well-controlled framework. Our main result will be to show that identities like (1.2) are equivalent to dynamical equations of motion in the closed string description.

The plan of the paper is as follows. In Section 2, we introduce some basic ideas on a very simple example and present the model we are studying, the  $\mathcal{N} = 1$  super Yang-Mills theory with gauge group  $U(N)$ , one adjoint chiral superfield and  $N_f \leq 2N$  flavours of quarks. We also state the chiral ring consistency theorem [8]. This is our main result and the proof of the theorem is given in Section 3. Finally in Section 4 we summarize our findings and conclude.

## 2 Preliminaries

### 2.1 A simple example: the classical limit

We can immediately give the flavour of the arguments that we are going to use by looking at the classical limit. We consider the  $U(N)$  super Yang-Mills theory with one adjoint chiral superfield  $X$  and tree-level superpotential  $\text{Tr } W(X)$  such that

$$W'(z) = \sum_{k=0}^d g_k z^k = g_d \prod_{i=1}^d (z - w_i). \quad (2.1)$$

The equations of motion in the open string description are thus

$$W'(X) = 0. \quad (2.2)$$

The most general solution is labeled by the positive integers  $N_i$ , with

$$\sum_{i=1}^d N_i = N, \quad (2.3)$$

such that the matrix  $X$  has  $N_i$  eigenvalues equal to  $w_i$ . In particular, the generating function

$$R(z) = \text{Tr} \frac{1}{z - X} = \sum_{k \geq 0} \frac{u_k}{z^{k+1}} \quad (2.4)$$

is given by

$$R(z) = \sum_i \frac{N_i}{z - w_i}. \quad (2.5)$$

In the closed string description, we can use only the gauge invariant operators  $u_k$ , not the matrix  $X$ . The equations of motion (2.2) are then written as

$$\text{Tr}(X^{n+1}W'(X)) = 0 = \sum_{k \geq 0} g_k u_{n+k+1}, \quad n \geq -1. \quad (2.6)$$

In terms of  $R(z)$ , this is equivalent to the existence of a degree  $d - 1$  polynomial  $\Delta$  such that

$$W'(z)R(z) = \Delta(z). \quad (2.7)$$

The vanishing of the terms proportional to negative powers of  $z$  in the large  $z$  expansion of the left hand side of (2.7) is indeed equivalent to the equations (2.6). The most general solution to (2.6) or (2.7) is given by

$$R(z) = \frac{\Delta(z)}{W'(z)} = \sum_{i=1}^d \frac{c_i}{z - w_i}. \quad (2.8)$$

The constants  $c_i$  can be arbitrary complex numbers, with the only constraint

$$\sum_{i=1}^d c_i = N \quad (2.9)$$

that follows from the definition of  $R(z)$ .

To make contact with the open string formula (2.5), we have to prove that the  $c_i$  must be positive integers. This is obvious in the open string framework since  $c_i = N_i$  is then identified with the number of eigenvalues of the matrix  $X$  that are equal to  $w_i$ . The question is: how can we understand this quantization condition in a formulation where only the gauge invariant operators  $u_k$  are available?

The fundamental idea is to implement the constraints (1.2) [8]. We are going to show the simple

**Theorem.** *The equations (2.6) are consistent with the constraints (1.2) if and only if the constants  $c_i$ s in (2.8) are positive integers. In particular, the integrals  $\frac{1}{2i\pi} \oint R dz$  over any closed contours are integers.*

This is a toy version of the chiral ring consistency theorem that we shall prove later. Very concretely, it means that a set of variables  $u_k$  given by the formulas

$$u_k = \sum_{i=1}^d c_i w_i^k \quad (2.10)$$

can satisfy the constraints (1.2) if and only if the  $c_i$ s are positive integers. To prove this simple algebraic result, we use the following trick. We introduce the function  $F(z)$  defined by the conditions

$$\frac{F'(z)}{F(z)} = R(z), \quad F(z) \underset{z \rightarrow \infty}{\sim} z^N. \quad (2.11)$$

In terms of the matrix  $X$ , one would simply have  $F(z) = \det(z - X)$ , but we do not want to use the matrix  $X$  here but only deal with the gauge invariant variables  $u_k$ . The function  $F$  is expressed in terms of these variables by integrating (2.4),

$$F(z) = z^N \exp\left(-\sum_{k \geq 1} \frac{u_k}{k z^k}\right). \quad (2.12)$$

The crucial algebraic property is that the relations (1.2) are *equivalent* to the fact that  $F(z)$  is a polynomial. A very effective way to compute the polynomials  $P_p$  is actually to write that the terms with a negative power of  $z$  in the large  $z$  expansion of the right-hand side of (2.12) must vanish. If  $F$  is a polynomial, then of course it is a single-valued function of  $z$ , and thus

$$\frac{1}{2i\pi} \oint R dz = \frac{1}{2i\pi} \oint d \ln F \in \mathbb{Z}. \quad (2.13)$$

In particular, the  $c_i$ s are integers. They are positive because  $F$  does not have poles. Conversely, if the  $c_i$  are positive integers, then we can introduce the matrix  $X$  defined to have  $c_i$  eigenvalues equal to  $w_i$  for all  $i$ . The relations (1.2) are then automatically satisfied.

## 2.2 The model

Our aim in the present paper is to generalize the above analysis to the full non-perturbative quantum theory, by analysing the consistency between the quantum versions of (2.6) and (1.2) to prove that the periods  $\frac{1}{2i\pi} \oint R dz$  must always be quantized. These quantization conditions are highly non-trivial constraints, known to be equivalent to a specific form of the Dijkgraaf-Vafa glueball superpotential, including the Veneziano-Yankielowicz coupling-independent part, and to contain the crucial information on the non-perturbative dynamics of the theory in the matrix model formalism [9, 10, 8].

We shall focus on the  $U(N)$  theory with one adjoint chiral superfield  $X$  and  $N_f$  flavours of fundamentals  $(\tilde{Q}^a, Q_b)$ . We always assume that the theory is asymptotically free or conformal in the UV,

$$N_f \leq 2N. \quad (2.14)$$

When  $N_f < 2N$  the instanton factor is given by

$$q = \Lambda^{2N-N_f} \quad (2.15)$$

in terms of the dynamically generated complex scale  $\Lambda$ . When  $N_f = 2N$  we have

$$q = e^{-8\pi^2/g^2 + i\vartheta} \quad (2.16)$$

in terms of the Yang-Mills coupling constant  $g$  and  $\vartheta$  angle. The tree-level superpotential has the form

$$W_{\text{tree}} = \text{Tr } W(X) + \sum_{1 \leq a, b \leq N_f} {}^T \tilde{Q}^a m_a^b(X) Q_b. \quad (2.17)$$

The derivative of  $W(z)$  is as in (2.1), and  $m_a^b(z)$  is a  $N_f \times N_f$  matrix-valued polynomial,

$$m_a^b(z) = \sum_{k=0}^{\delta} m_{a,k}^b z^k, \quad (2.18)$$

with

$$\det m(z) = U(z) = U_0 \prod_{Q=1}^{N_f \delta} (z - b_Q). \quad (2.19)$$

It is useful to introduce the symmetric polynomials

$$\sigma_\alpha = \sum_{Q_1 < \dots < Q_\alpha} b_{Q_1} \cdots b_{Q_\alpha}, \quad 1 \leq \alpha \leq N_f \delta. \quad (2.20)$$

We shall consider the case where  $m_a^b(z)$  is a linear function of  $z$ ,

$$\delta = 1, \quad (2.21)$$

in the following.<sup>1</sup>

The classical theory has a large number of vacua obtained by extremizing the superpotential (2.17). The most general solution  $|N_i; \nu_Q\rangle_{\text{cl}}$  is labeled by the numbers of eigenvalues of the matrix  $X$ ,  $N_i \geq 0$  and  $\nu_Q = 0$  or 1, that are equal to  $w_i$  and  $b_Q$  respectively [9]. The constraint

$$\sum_{i=1}^d N_i + \sum_{Q=1}^{N_f} \nu_Q = N \quad (2.22)$$

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<sup>1</sup>This is not strictly necessary as long as the constraint  $N_f \delta \leq 2N$  is satisfied.

must be satisfied. The gauge group  $U(N)$  is broken down to  $U(N_1) \times \cdots \times U(N_d)$  in a vacuum  $|N_i; \nu_Q\rangle_{\text{cl}}$ . We shall call the number of non-zero integers  $N_i$  the *rank*  $r$  of the vacuum.

In addition to (1.1), we have other basic gauge invariant operators in the theory that are constructed by using the vector chiral superfield  $W^\alpha$ ,

$$u_k^\alpha = \frac{1}{4\pi} \text{Tr } W^\alpha X^k, \quad v_k = -\frac{1}{16\pi^2} \text{Tr } W^\alpha W_\alpha X^k, \quad w_{a,k}^b = {}^T \tilde{Q}^b X^k Q_a. \quad (2.23)$$

The associated generating functions are defined by

$$\mathcal{W}^\alpha(z) = \sum_{k \geq 0} \frac{u_k^\alpha}{z^{k+1}}, \quad S(z) = \sum_{k \geq 0} \frac{v_k}{z^{k+1}}, \quad G_a^b(z) = \sum_{k \geq 0} \frac{w_{a,k}^b}{z^{k+1}}. \quad (2.24)$$

The relations that replace (2.6), or equivalently (2.7), in the full quantum theory are given by the following generalized Konishi anomaly equations [11]

$$NW'(z)R(z) + Nm'_a{}^b(z)G_b^a(z) - 2S(z)R(z) - 2\mathcal{W}^\alpha(z)\mathcal{W}_\alpha(z) = \Delta_R(z) \quad (2.25)$$

$$NW'(z)\mathcal{W}^\alpha(z) - 2S(z)\mathcal{W}^\alpha(z) = \Delta^\alpha(z) \quad (2.26)$$

$$NW'(z)S(z) - S(z)^2 = \Delta_S(z) \quad (2.27)$$

$$NG_a^c(z)m_c^b(z) - S(z)\delta_a^b = \Delta_a^b(z) \quad (2.28)$$

$$Nm_a^c(z)G_c^b(z) - S(z)\delta_a^b = \tilde{\Delta}_a^b(z). \quad (2.29)$$

The functions  $\Delta_R$ ,  $\Delta^\alpha$ ,  $\Delta_S$ ,  $\Delta_a^b$  and  $\tilde{\Delta}_a^b$  must be polynomials. By expanding (2.25)–(2.29) at large  $z$ , and writing that the terms proportional to negative powers of  $z$  must vanish, we obtain an infinite set of constraints on the gauge invariant operators, valid for any integer  $n \geq -1$ ,

$$N \sum_{k \geq 0} (g_k u_{n+k+1} + (k+1)m_{a,k+1}^b w_{b,n+k+1}^a) - 2 \sum_{k_1+k_2=n} (u_{k_1} v_{k_2} + u_{k_1}^\alpha u_{k_2\alpha}) = 0 \quad (2.30)$$

$$N \sum_{k \geq 0} g_k u_{n+k+1}^\alpha - 2 \sum_{k_1+k_2=n} v_{k_1} u_{k_2}^\alpha = 0 \quad (2.31)$$

$$N \sum_{k \geq 0} g_k v_{n+k+1} - \sum_{k_1+k_2=n} v_{k_1} v_{k_2} = 0 \quad (2.32)$$

$$N \sum_{k \geq 0} w_{a,k+n+1}^c m_{c,k}^b - v_{n+1} \delta_a^b = 0 \quad (2.33)$$

$$N \sum_{k \geq 0} m_{a,k}^c w_{c,k+n+1}^b - v_{n+1} \delta_a^b = 0. \quad (2.34)$$

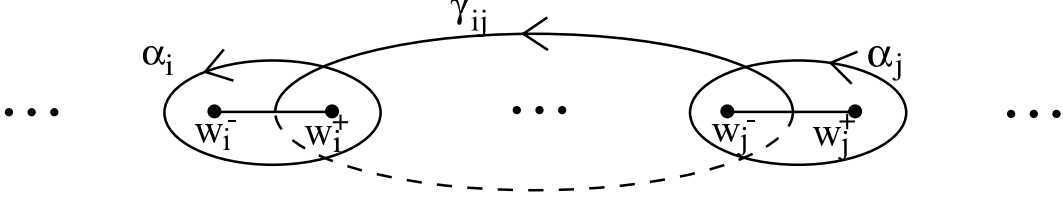


Figure 1: The hyperelliptic Riemann surface  $\mathcal{C}$ , with the contours  $\alpha_i$  and  $\gamma_{ij}$  used in the main text.

Equations (2.25)–(2.29) show that the generating functions are meromorphic functions on a hyperelliptic Riemann surface of the form

$$\mathcal{C}_r : y_r^2 = \prod_{i=1}^r (z - w_i^-)(z - w_i^+). \quad (2.35)$$

The integer  $r$ , called the rank of the solution, must satisfy

$$r \leq d \quad (2.36)$$

and we have

$$W'(z)^2 - 4\Delta_{d-1}(z) = \phi_{d-r}(z)^2 y_r^2 \quad (2.37)$$

for some polynomials  $\Delta_{d-1} = \Delta_S/N^2$  and  $\phi_{d-r}$  of degrees  $d-1$  and  $d-r$  respectively. The curve (2.35) with some closed contours is depicted in Figure 1. It corresponds to the geometry of the closed string background. We shall use extensively in the following the most general solution to (2.25)–(2.29) of rank  $r$  for the expectation value  $\langle R(z) \rangle_r$ . It has the form [9]

$$\langle R(z) \rangle_r = \frac{C_{r-1}}{y_r} + \frac{1}{2} \frac{U'}{U} - \frac{1}{2y_r} \sum_Q \frac{(1 - 2\nu_Q)y_r(z = b_Q)}{z - b_Q}. \quad (2.38)$$

The polynomial  $C_{r-1} = \frac{1}{2}(2N - N_f)z^{r-1} + \dots$  is of degree  $r-1$  and is a priori unknown except for its term of highest degree that is fixed by the large  $z$  asymptotics of  $R(z)$ . In the classical limit, the solutions (2.38) correspond to the rank  $r$  classical vacua  $|N_i; \nu_Q\rangle_{\text{cl}}$  described previously. Quantum mechanically, the anomaly equations (2.38) leave  $2r-1$  arbitrary parameters, which are the coefficients of  $C_{r-1}$  and of  $\Delta_{d-1}$  that are not fixed by the factorization condition (2.37). These unknown parameters are the quantum analogues of the coefficients  $c_i$  in (2.8), and our main goal is to show that they are fixed by a quantum version of the simple consistency proof explained in 2.1.



## 2.3 Chiral ring relations and anomaly equations

The model (2.17) has a useful  $SU(N_f)_L \times SU(N_f)_R \times U(1)_A \times U(1)_B \times U(1)_R$  global symmetry. The charges of the various parameters and operators of the theory are given in the following table

	$u_k$	$u_k^\alpha$	$v_k$	$w_{a,k}^b$	$g_k$	$m_{a,k}^b$	$\sigma_k$	$U_0$	$q$
$U(1)_A$	$k$	$k$	$k$	$k-1$	$-k-1$	$-k+1$	$k$	0	$2N - N_f$
$U(1)_B$	0	0	0	2	0	-2	0	$-2N_f$	$2N_f$
$U(1)_R$	0	1	2	2	2	0	0	0	0
$SU(N_f)_L$	<b>1</b>	<b>1</b>	<b>1</b>	<b><math>N_f</math></b>	<b>1</b>	<b><math>N_f</math></b>	<b>1</b>	<b>1</b>	<b>1</b>
$SU(N_f)_R$	<b>1</b>	<b>1</b>	<b>1</b>	<b><math>\overline{N_f}</math></b>	<b>1</b>	<b><math>\overline{N_f}</math></b>	<b>1</b>	<b>1</b>	<b>1</b>

(2.39)

It is useful for our purposes to consider the subring  $\mathcal{A}_0$  of the chiral ring of the theory that is invariant under  $SU(N_f)_L \times SU(N_f)_R \times U(1)_B \times U(1)_R$ . This subring is generated by the operators  $u_k$  and the parameters<sup>2</sup>  $\sigma_k$  and

$$\mathbf{q} = U_0 q. \quad (2.40)$$

It is a simple polynomial ring given by

$$\mathcal{A}_0 = \mathbb{C}[\mathbf{q}, \sigma_1, \dots, \sigma_{N_f}, u_1, \dots, u_N]. \quad (2.41)$$

As stressed in the Section 2 of [8], a polynomial ring has no deformation, and thus (2.41), which is trivially valid at the classical level due to the relations (1.2), is also valid in the full quantum theory. The meaning of this statement is simply that any operator in  $\mathcal{A}_0$  can be expressed as a finite sum of finite products of  $U_0 q$ ,  $\sigma_k$  for  $1 \leq k \leq N_f$  and  $u_k$  for  $1 \leq k \leq N$ , a rather trivial result. It is sometimes claimed in the literature that the ring  $\mathcal{A}_0$  is deformed because the relations (1.2) can get quantum corrections. This is not correct. In the full quantum theory, we can *define* what is meant by  $u_k \in \mathcal{A}_0$  for  $k > N$  by a relation of the form

$$u_{N+p} = \mathcal{P}_p(u_1, \dots, u_N, \sigma_1, \dots, \sigma_{N_f}, \mathbf{q}), \quad p \geq 1. \quad (2.42)$$

The  $\mathcal{P}_p$  are chosen to be consistent with the symmetries (2.39) and the classical limit (1.2), but can be completely arbitrary otherwise. It can be convenient to work with a particular definition (2.42), and we shall see shortly that there is indeed a canonical choice, but this remains a choice and has no physical content [8]. Let us note that a parallel discussion applies to the variables  $u_k^\alpha$ ,  $v_k$  and  $w_{a,k}^b$ , which are independent only for  $0 \leq k \leq N-1$ .

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<sup>2</sup>It is convenient to include the parameters, which can always be promoted to background chiral superfields, in the chiral ring.

The equations (2.25)–(2.29) were derived in perturbation theory in [12, 11]. At the non-perturbative level, these equations *do* get quantum corrections. However, these quantum corrections have a very special form. The general theorem is as follows:

**Non-perturbative anomaly theorem** [6]: *The non-perturbative corrections to the generalized anomaly equations are such that they can be absorbed in a non-perturbative redefinition of the variables that enter the equations.*

This means that there exists a canonical choice for the definitions of the variables  $u_k$  for  $k > N$  as in (2.42), and other similar canonical definitions of the variables  $u_{k-1}^\alpha$ ,  $v_{k-1}$  and  $w_{a,k-1}^b$  for  $k > N$ , that make all the non-perturbative corrections *implicit*. The theorem has been proven recently in the case of the theory with no flavour [6]. The arguments used in [6] can in principle be generalized straightforwardly, and we shall take the result for granted in the theory with flavours as well.

## 2.4 The chiral ring consistency theorem

We can now state the quantum version of the classical problem solved in Section 2.1. On the one hand, in the closed string description, the theory is described by the equations (2.25)–(2.29) or equivalently by (2.30)–(2.34). On the other hand, we know that the existence of the open string formulation implies that relations of the form (2.42) must exist. These relations imply that there are only a finite number of independent variables. The anomaly equations (2.30)–(2.34) thus yield an infinite set of constraints on a finite set of independent variables. Generically, such an overconstrained system of equations is inconsistent. The main result of the present work is to prove the

**Chiral ring consistency theorem:** *The system of equations (2.30)–(2.34) is consistent with the existence of relations of the form (2.42) if and only if the periods of the gauge theory resolvent  $\frac{1}{2i\pi} \oint R dz$  are integers. The relations (2.42) (and all the other relations amongst chiral operators) are then fixed in a unique way.*

This theorem was conjectured in [8]. As discussed in 2.2, the equations (2.30)–(2.34) imply that  $R$  is a meromorphic function on a hyperelliptic curve of the form (2.35). The theorem then states that the algebraic consistency of the chiral ring implies

$$\frac{1}{2i\pi} \oint_{\alpha_i} \langle R \rangle_r dz \in \mathbb{Z} \quad (2.43)$$

$$\frac{1}{2i\pi} \oint_{\gamma_{ij}} \langle R \rangle_r dz \in \mathbb{Z}, \quad (2.44)$$

where the contours  $\alpha_i$  and  $\gamma_{ij}$  are defined in Figure 1. Actually, the  $\alpha_i$ -periods are automatically positive, as we shall see. Several comments on this result are in order.

First, the equations (2.43) and (2.44) yield  $2r - 2$  non-trivial constraints on (2.38) (the sum of the equations (2.43) is trivial because the asymptotic condition

$$\langle R(z) \rangle \underset{z \rightarrow \infty}{\sim} \frac{N}{z} \quad (2.45)$$

is automatically satisfied by (2.38)). Thus the solution is uniquely fixed, up to a single unknown that can be identified with the quantum deformation parameter. We shall explain in 3.4 how to relate precisely this parameter to the instanton factor  $q$ .

The quantization conditions (2.43) are the quantum versions of the classical result (2.13). Note that it is not correct to claim that (2.43) is obvious because the period integral yields the number of eigenvalues of the matrix  $X$  in the cut  $[w_i^-, w_i^+]$ . This interpretation is completely erroneous in the context of the finite  $N$  gauge theory [8]. Actually, most of the possible definitions (2.42) would violate (2.43). The correct interpretation of equation (2.43) is that it yields non-trivial constraints on the canonical definitions of the variables for which the anomaly equations have the simple form (2.30)–(2.34), in line with the non-perturbative anomaly theorem in Section 2.3.

The quantization conditions (2.44) have no classical counterpart. In the closed string formulation of the theory, three-form fluxes are turned on. The associated flux superpotential coincides with the Dijkgraaf-Vafa glueball superpotential  $W_{\text{DV}}(S_i)$ , and the equations (2.44) are equivalent to the extremization of  $W_{\text{DV}}$  [9]. The chiral ring consistency theorem thus answers, for the chiral sector of the theory, the questions asked in Section 1: the existence of the relations (2.42), which are trivial off-shell identities in the open string description, are seen in the closed string formulation only after implementing the closed string dynamical equations of motion. The exchange of off-shell identities and on-shell dynamical equations in the open/closed string duality was emphasized in [6].

Another important consequence of the theorem is to lift the mystery of the Veneziano-Yankielowicz term  $f(S_i)$  in  $W_{\text{DV}}$ . It was shown in [12] that an arbitrary function  $f(S_i)$ , depending on the glueball superfields  $S_i$  but independent of the couplings in the tree-level superpotential (2.17), could be added to  $W_{\text{DV}}$  without spoiling the correspondence with the matrix model. The term  $f(S_i)$  plays of course a crucial rôle in fixing the on-shell values of the glueballs, and is at the heart of the non-perturbative gauge dynamics. However, it is left unconstrained by the anomaly equations (2.25)–(2.29), whose most general solutions are simply parametrized by the  $S_i$ . From the point of view of the matrix model, the glueballs  $S_i$  are identified with the filling fractions which are completely arbitrary parameters. For these reasons, and as discussed at length in [12] for example, the determination from first principles of the function  $f(S_i)$  seemed to be out of reach. We now see that the situation is con-

ceptually must simpler than what might have been expected [8]: the filling fractions, and thus the Veneziano-Yankielowicz term  $f(S_i)$ , are fixed entirely by imposing the consistency between (2.25)–(2.29) and (2.42). The fact that this term ought to be fixed by general consistency conditions was first emphasized in [10].

To prove our main theorem, we are going to show that the function  $F$  defined by (2.11) must satisfy the fundamental equation

$$F(z) + \frac{qU(z)}{F(z)} = H(z) \quad (2.46)$$

for a polynomial  $H = (1 + \mathbf{q}\delta_{N_f, 2N})z^N + \dots$  of degree  $N$ . In the classical theory  $q = 0$ , (2.46) simply says that  $F$  must be a polynomial, and we have explained after (2.12) that this condition is equivalent to the relations (1.2). Similarly, in the quantum theory, (2.46) is *equivalent* to a particular quantum corrected form

$$u_{N+p} = \mathcal{P}_p^{(0)}(u_1, \dots, u_N, \sigma_1, \dots, \sigma_{N_f}, \mathbf{q}), \quad p \geq 1, \quad (2.47)$$

for the relations (2.42) [8]. This result is obtained straightforwardly by expanding the left hand side of (2.46) at large  $z$ .

Equation (2.46) implies that

$$F = \frac{1}{2} \left( H + \sqrt{H^2 - 4qU} \right) \quad (2.48)$$

is a meromorphic function on the hyperelliptic surface

$$\tilde{\mathcal{C}} : Y^2 = H(z)^2 - 4qU(z). \quad (2.49)$$

The generating function

$$R = \frac{F'}{F} = \frac{1}{2} \frac{U'}{U} + \left( H' - \frac{U'H}{2U} \right) \frac{1}{\sqrt{H^2 - 4qU}} \quad (2.50)$$

is then also automatically a meromorphic function on the same curve  $\tilde{\mathcal{C}}$ . From the single-valuedness of  $F$  on  $\tilde{\mathcal{C}}$ , we deduce that

$$\frac{1}{2i\pi} \oint_c R dz = \frac{1}{2i\pi} \oint_c d \ln F \in \mathbb{Z} \quad (2.51)$$

for any closed contour  $c$ . Note that the consistency of (2.50) with the fact that  $\langle R \rangle_r$  must be well-defined on the curve (2.35) implies that the following factorization condition must hold in the rank  $r$  vacua

$$\langle H(z) \rangle_r^2 - 4qU(z) = \psi_{N-r}(z)^2 y_r^2, \quad (2.52)$$

for some degree  $N - r$  polynomial  $\psi_{N-r}$ . The equations (2.51) thus automatically imply (2.43) and (2.44). The positivity of the  $\alpha_i$ -periods is a direct consequence of the classical limit. We shall focus on proving (2.46) in the following.

### 3 The proof of the main theorem

#### 3.1 Generalities

We suppose from now on that  $N_f = 2N$ . The other cases with  $N_f < 2N$  can be obtained by integrating out some flavours, sending their masses to infinity. If not explicitly stated otherwise, we shall always assume that the degree of  $W'$  in (2.1) is

$$d = N. \quad (3.1)$$

This is not a restriction, because the  $U(1)_R$  symmetry implies that the relations (2.42) we want to study cannot depend on the couplings  $g_k$  in  $W$ .

It is convenient to define new variables  $x_1, \dots, x_N$  by the relations

$$u_k = \sum_{i=1}^N x_i^k \quad \text{for } 1 \leq k \leq N. \quad (3.2)$$

Strictly speaking, the  $x_i$ s are not in the chiral subring  $\mathcal{A}_0$ , but they can always be introduced by using the following algebraic trick. We consider the polynomial part  $F_0$  of the function  $F$  at large  $z$ ,

$$F(z) = F_0(z) + \mathcal{O}(1/z). \quad (3.3)$$

Equation (2.12) shows that the coefficients of  $F_0$  are themselves polynomials in the  $u_k$ , and thus  $F_0 \in \mathcal{A}_0[T]$  where  $T$  is an undeterminate. It is then trivial to check that the  $x_i$  satisfying (3.2) are the roots of the polynomial  $F_0$  in its splitting field.<sup>3</sup>

We can use the new variables to rewrite the relations (2.42) in the form

$$u_{N+p} = \mathcal{P}_p(\mathbf{x}, \mathbf{b}, \mathbf{q}). \quad (3.4)$$

We use boldface letters to represent collectively a set of variables, for example  $\mathbf{x}$  represents all the  $x_i$ ,  $1 \leq i \leq N$ . In (3.4),  $\mathcal{P}_p$  must be invariant under the action of the permutation group  $S_N \times S_{2N}$  that act on the  $x_i$ s and the  $b_Q$ s independently.<sup>4</sup>

Let us introduce the vector spaces  $\mathcal{V}_k$  of arbitrary power series in  $\mathbf{x}$ ,  $\mathbf{b}$  and  $\mathbf{q}$  that are invariant under the action of  $S_N \times S_{2N}$  and that are homogeneous of degree  $k$ , the degree being identified with the A-charge defined by (2.39). Clearly,  $\mathcal{P}_p \in \mathcal{V}_{N+p}$ .

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<sup>3</sup>The existence of the splitting field for any polynomial and thus of the variables  $x_i$  is ensured by standard theorems in elementary algebra, see for example [13].

<sup>4</sup>The  $\mathcal{P}_p$  appearing in (2.42) and (3.4) are of course not the same. We use the same notation because they coincide when the relations (3.2) and (2.20) are taken into account.

Since  $\mathbf{x}$  and  $\mathbf{b}$  are of degree one, elements of  $\mathcal{V}_k$  must be polynomials in the  $x_i$ s and  $b_{Qs}$ . On the other hand,  $\mathbf{q}$  is of degree  $2N - N_f$ . In the general case  $N_f = 2N$  we are considering, arbitrary powers of  $\mathbf{q}$  can thus in principle appear.

The equations (3.4) are operator equations. Taking the expectation value, we get

$$\langle u_{N+p} \rangle = \mathcal{P}_p(\langle \mathbf{x} \rangle, \mathbf{b}, \mathbf{q}), \quad (3.5)$$

in *all* the vacua of the theory. Of course,  $\langle u_{N+p} \rangle$  and  $\langle \mathbf{x} \rangle$  depend on the particular vacua under consideration, but the polynomials  $\mathcal{P}_p$  do *not*. With this constraint, it is easy to realize that the equations (3.5) cannot be consistent with the most general solution (2.38) to the anomaly equations. We are going to prove that consistency is achieved only when (2.46) is satisfied, which corresponds to the quantization conditions (2.43) and (2.44) and to the particular form (2.47)

$$\mathcal{P}_p = \mathcal{P}_p^{(0)} \quad (3.6)$$

of the relations (3.5).

We shall use the following strategy. The solutions (2.38) are uniquely fixed for the rank zero vacua. We are going to show that this implies that the polynomial  $\mathcal{P}_1$  must be equal to  $\mathcal{P}_1^{(0)}$ . But this provides a non-trivial operator constraint, that fixes uniquely the solutions (2.38) at rank one. Analysing the form of these solutions, we can then show that (3.6) or (2.47) must be valid at least for  $1 \leq p \leq N + 5$ . This yields  $N + 5$  operator constraints, that can be used to fix the solutions (2.38) for  $2r - 1 \leq N + 5$ . In particular, we know the rank two solutions for any  $N$ . This turns out to imply that (2.47) must be true at least for  $p \leq 2N + 7$ . This yields a total of  $2N + 7$  constraints, a number greater than the maximum number of unknown parameters  $2N - 1$  that can appear in (2.38). We can then check that all the resulting solutions do satisfy (2.47) for all  $p$ .

## 3.2 Using the rank zero vacua

Let us start by looking at the vacua of rank zero, that correspond to a completely broken gauge group. There is no free parameter in this case, and thus the solution must be completely fixed. This is not difficult to check. The factorization condition (2.37) yields when  $r = 0$

$$4\Delta_{N-1} = W'^2 - \phi_N^2 = (W' - \phi_N)(W' + \phi_N). \quad (3.7)$$

Since  $W'$  and  $\phi_N$  are of degree  $N$ , whereas  $\Delta_{N-1}$  is of degree  $N - 1$ , (3.7) implies that  $\phi_N = \pm W'$  and  $\Delta_{N-1} = 0$ . Equation (2.27) then shows that  $\langle S(z) \rangle_{r=0} = 0$ . From

the other anomaly equations, we immediately derive that there can be no quantum correction at all. In particular,

$$\langle R(z) \rangle_{r=0} = \sum_{Q=1}^{2N} \frac{\nu_Q}{z - b_Q}, \quad \langle F(z) \rangle_{r=0} = \prod_{Q=1}^{2N} (z - b_Q)^{\nu_Q}. \quad (3.8)$$

We can thus compute

$$\langle F(z) \rangle_{r=0} + \frac{qU(z)}{\langle F(z) \rangle_{r=0}} = \prod_{Q=1}^{2N} (z - b_Q)^{\nu_Q} + \mathbf{q} \prod_{Q=1}^{2N} (z - b_Q)^{1-\nu_Q}, \quad (3.9)$$

which is indeed a polynomial, consistently with (2.46). This shows that

$$\langle u_{N+p} \rangle_{r=0} = \mathcal{P}_p^{(0)}(\langle \mathbf{x} \rangle_{r=0}, \mathbf{b}, \mathbf{q}), \quad p \geq 1, \quad (3.10)$$

in the vacua of rank zero.

What can we learn from (3.10) on the possible forms of the operator relations (3.4)? Let us decompose  $\mathcal{P}_p$  as the sum of two terms,

$$u_{N+p} = \mathcal{P}_p^{(0)}(\mathbf{x}, \mathbf{b}, \mathbf{q}) + \mathcal{P}_p^{(1)}(\mathbf{x}, \mathbf{b}, \mathbf{q}). \quad (3.11)$$

Equation (3.10) is equivalent to the constraints

$$\mathcal{P}_p^{(1)}(\langle \mathbf{x} \rangle_{r=0}, \mathbf{b}, \mathbf{q}) = 0, \quad p \geq 1, \quad (3.12)$$

where the expectation values are taken in the rank zero vacua only. It is extremely important to understand that this constraint does *not* imply that  $\mathcal{P}_p^{(1)} = 0$ , because the variables  $x_i$  and  $b_j$  are not algebraically independent in the rank zero vacua. This subtlety is completely general. When one focuses on a special set of vacua, operator relations can only be determined modulo the ideal generated by the exceptional chiral ring relations that are valid only in the particular vacua under consideration. It is clear that there are many such relations in the rank zero vacua. Actually, the fact that there are no quantum corrections in (3.8) ensures that the classical relations (1.2) must be valid,

$$\langle u_{N+p} \rangle_{r=0} = P_p(\langle \mathbf{x} \rangle_{r=0}) \quad p \geq 1. \quad (3.13)$$

Consistency with (3.10) implies that

$$\mathcal{P}_p^{(0)}(\langle \mathbf{x} \rangle_{r=0}, \mathbf{b}, \mathbf{q}) - P_p(\langle \mathbf{x} \rangle_{r=0}) = 0. \quad (3.14)$$

The relations (3.14) prevent an analysis based only on the rank zero vacua to fix unambiguously the operator relations (3.11).

Even though (3.12) does not show that  $\mathcal{P}_p^{(1)}$  must vanish for all  $p$ , it does put some non-trivial constraints. In the rank zero vacua, (3.8) shows that the set  $\{x_1, \dots, x_N\}$  is identified with a subset of  $\{b_1, \dots, b_{2N}\}$ . Taking into account the fact that  $\mathcal{P}^{(1)}$  is symmetric under permutation of the  $b_i$ s, we deduce that (3.12) is equivalent to

$$\mathcal{P}_p^{(1)}(x_1 = b_1, x_2 = b_2, \dots, x_N = b_N, \mathbf{b}, \mathbf{q}) = 0. \quad (3.15)$$

Let us now show the following

**Proposition:** *Let  $A \in \mathcal{V}_n$ ,  $A \neq 0$ , such that*

$$A(x_1 = b_1, x_2 = b_2, \dots, x_N = b_N, \mathbf{b}, \mathbf{q}) = 0. \quad (3.16)$$

*Then  $\deg A = n \geq N + 1$ . Moreover, if  $\deg A = N + 1$ , then  $A$  must be of the form*

$$A(\mathbf{x}, \mathbf{b}, \mathbf{q}) = \mathbf{a}(\mathbf{q}) \sum_{i=1}^N \frac{\prod_{Q=1}^{2N} (x_i - b_Q)}{\prod_{j \neq i} (x_i - x_j)}, \quad (3.17)$$

*for some  $\mathbf{x}$ - and  $\mathbf{b}$ -independent power series  $\mathbf{a}$  in  $\mathbf{q}$ .*

To prove the proposition, we note that (3.16) ensures that there exists  $k_0 \leq N$ , defined to be the smallest integer such that  $A(x_1 = b_1, \dots, x_{k_0} = b_{k_0}, x_{k_0+1}, \dots, x_N, \mathbf{b}, \mathbf{q})$  identically vanishes (for all  $x_{k_0+1}, \dots, x_N$ ). By symmetry of the  $b_i$ s, we have

$$A(x_1 = b_1, \dots, x_{k_0-1} = b_{k_0-1}, x_{k_0} = b_k, x_{k_0+1}, \dots, x_N, \mathbf{b}, \mathbf{q}) = 0 \text{ for all } k \geq k_0. \quad (3.18)$$

Seeing  $A(x_1 = b_1, \dots, x_{k_0-1} = b_{k_0-1}, x_{k_0}, \dots, x_N, \mathbf{b}, \mathbf{q})$  as a polynomial of a single variable  $x_{k_0}$ , (3.18) implies that

$$A(x_1 = b_1, \dots, x_{k_0-1} = b_{k_0-1}, x_{k_0}, \dots, x_N, \mathbf{b}, \mathbf{q}) = \prod_{k=k_0}^{2N} (x_{k_0} - b_k) B \quad (3.19)$$

for some non-zero  $B$  ( $B = 0$  would contradict the defining property of  $k_0$ ). In particular,

$$\deg A(x_1 = b_1, \dots, x_{k_0-1} = b_{k_0-1}, x_{k_0}, \dots, x_N, \mathbf{b}, \mathbf{q}) \geq 2N - k_0 + 1. \quad (3.20)$$

Because  $A(x_1 = b_1, \dots, x_{k_0-1} = b_{k_0-1}, x_{k_0}, \dots, x_N, \mathbf{b}, \mathbf{q})$  does not vanish, and  $A$  is homogeneous, we must have  $\deg A = \deg A(x_1 = b_1, \dots, x_{k_0-1} = b_{k_0-1}, x_{k_0}, \dots, x_N, \mathbf{b}, \mathbf{q})$  and thus  $\deg A \geq 2N - k_0 + 1$ . Since  $k_0 \leq N$ , we get  $\deg A \geq N + 1$ , and the first part of the proposition is proven.

It is not difficult to construct polynomials satisfying (3.16) for any degree  $\geq N + 1$  using the following trick. We consider the rational function

$$\rho(z) = \frac{\prod_{Q=1}^{2N} (z - b_Q)}{\prod_{i=1}^N (z - x_i)}. \quad (3.21)$$



At large  $z$ , we can expand

$$\rho(z) = z^N + \sum_{n \geq 1} A_n(\mathbf{x}, \mathbf{b}) z^{N-n} \quad (3.22)$$

in terms of  $A_n \in \mathcal{V}_n$ . Clearly,  $\rho$  is a polynomial if and only if the set  $\{x_1, \dots, x_N\}$  is included in the set  $\{b_1, \dots, b_{2N}\}$ . This is true if and only if all the terms with a negative power of  $z$  in the expansion (3.22) vanish, and thus the polynomials  $A_n$  satisfy the constraint (3.16) for  $n \geq N + 1$ . It is easy to check that the polynomial  $A_{N+1}$  is proportional to the right-hand side of (3.17). To show the uniqueness of the solution at degree  $N + 1$ , we note that (3.19) implies that the coefficient of  $\sum_i x_i^{N+1}$  in a non-vanishing solution of degree  $N + 1$  must be non-zero. If we have two non-zero solutions, we can always consider a linear combination for which the terms in  $\sum_i x_i^{N+1}$  cancel. The linear combination must then vanish, showing that the two solutions we started with are proportional to each other. This ends the proof of the proposition.

Using (3.15), we can apply the proposition to  $A = \mathcal{P}_p^{(1)}$ . First, it shows that either  $\mathcal{P}_p^{(1)} = 0$  or  $\deg \mathcal{P}_p^{(1)} \geq N + 1$ . In the present case, this is quite useless because  $\deg \mathcal{P}_p^{(1)} = N + p \geq N + 1$  is true by construction. However, similar non-trivial inequalities will be of great help in the next two subsections. Second, the proposition implies that

$$u_{N+1} = \mathcal{P}_1(\mathbf{x}, \mathbf{b}, \mathbf{q}) = \sum_{i=1}^N x_i^{N+1} + \mathbf{a}(\mathbf{q}) \sum_{i=1}^N \frac{\prod_{Q=1}^{2N} (x_i - b_Q)}{\prod_{j \neq i} (x_i - x_j)}, \quad (3.23)$$

for some a priori unknown series  $\mathbf{a}(\mathbf{q}) = a_1 \mathbf{q} + \dots$  in  $\mathbf{q}$ . There are two possible attitudes with regard to the function  $\mathbf{a}(\mathbf{q})$ . A first possibility is to consider  $\mathbf{a}$  to be the quantum deformation parameter instead of  $\mathbf{q}$ . In particular, expressing the results in terms of  $\mathbf{a}$  instead of  $\mathbf{q}$  is irrelevant for the proof of the chiral ring consistency theorem and of (2.43) and (2.44). A second possibility is to insist on using the instanton factor  $\mathbf{q}$ . It will be explained in subsection 3.4 how to prove that

$$\mathbf{a}(\mathbf{q}) = (N + 1) \frac{\mathbf{q}}{1 - \mathbf{q}}. \quad (3.24)$$

It is straightforward to check that (3.24) is consistent with (2.46) and thus equivalent to (2.47) for  $p = 1$ ,  $\mathcal{P}_1 = \mathcal{P}_1^{(0)}$ .

### 3.3 A useful lemma

To proceed further, we need a simple algebraic

**Lemma:** *Let  $n$  be a positive integer. Let  $|i\rangle$ ,  $i \in I$ , a subset of vacua, with classical*

limits  $|i\rangle_{\text{cl}}$ . Assume that we can prove, for all  $A \in \mathcal{V}_n$ , that  $A(\langle i|\mathbf{x}|i\rangle_{\text{cl}}, \mathbf{b}, \mathbf{q}) = 0$  for all  $i$  implies that  $A$  identically vanishes. Then if  $P \in \mathcal{V}_n$  is such that  $P(\langle i|\mathbf{x}|i\rangle, \mathbf{b}, \mathbf{q}) = 0$  for all  $i$ ,  $P$  must identically vanish.

This result is very useful, because the classical expectation values  $\langle i|\mathbf{x}|i\rangle_{\text{cl}}$  are much simpler than their quantum counterparts  $\langle i|\mathbf{x}|i\rangle$ .

To prove the lemma, we consider  $P \in \mathcal{V}_n$  such that

$$P(\langle i|\mathbf{x}|i\rangle, \mathbf{b}, \mathbf{q}) = 0 \quad (3.25)$$

for all  $i \in I$ . We expand

$$P(\mathbf{x}, \mathbf{b}, \mathbf{q}) = \sum_{k \geq 0} A_k(\mathbf{x}, \mathbf{b}) \mathbf{q}^k, \quad (3.26)$$

where  $A_k \in \mathcal{V}_n$ . Equation (3.25) is equivalent to

$$\sum_{k \geq 0} A_k(\langle i|\mathbf{x}|i\rangle, \mathbf{b}) \mathbf{q}^k = 0. \quad (3.27)$$

Note that, of course,  $\langle i|\mathbf{x}|i\rangle$  depends on  $\mathbf{q}$  in general.

Let us show recursively on  $k$  that (3.27) implies

$$A_k = 0 \quad (3.28)$$

for all  $k \geq 0$ . The vanishing of  $P$  will follow immediately. To prove the case  $k = 0$ , let us take the  $\mathbf{q} \rightarrow 0$  limit of (3.27),

$$A_0(\langle i|\mathbf{x}|i\rangle_{\text{cl}}, \mathbf{b}) = 0. \quad (3.29)$$

The vanishing of  $A_0$  then follows from the basic assumption in the lemma. Assume now that (3.28) is valid for  $k \leq k_0$ . Equation (3.27) then yields

$$\sum_{k \geq k_0+1} A_k(\langle i|\mathbf{x}|i\rangle, \mathbf{b}) \mathbf{q}^k = 0, \quad (3.30)$$

which implies that

$$A_{k_0+1}(\langle i|\mathbf{x}|i\rangle, \mathbf{b}) + \sum_{k \geq 1} A_{k_0+1+k}(\langle i|\mathbf{x}|i\rangle, \mathbf{b}) \mathbf{q}^k = 0. \quad (3.31)$$

We deduce that (3.28) is valid for  $k = k_0 + 1$  by taking the  $\mathbf{q} \rightarrow 0$  limit and applying again the basic assumption in the lemma.

### 3.4 Using the rank one vacua

For the vacua of rank one, the curve (2.35) is a sphere and  $\langle F(z) \rangle_{r=1}$ , which is obtained from  $\langle R(z) \rangle_{r=1}$  by performing elementary integrals, automatically satisfies a degree two algebraic equation. The solution is parametrized by a single unknown parameter, the glueball expectation value  $\langle S \rangle_{r=1} = \langle v_0 \rangle_{r=1}/N$ . It is completely elementary to check that

$$\langle F(z) \rangle_{r=1} + \frac{hU(z)}{\langle F(z) \rangle_{r=1}} = \langle H(z) \rangle_{r=1} \quad (3.32)$$

for some polynomial  $H$  and some function  $h$  of  $\langle S \rangle_{r=1}$  and of the parameters. For example, if we use a quadratic superpotential  $W$ , i.e.  $g_k = 0$  for  $k \geq 2$  (this is not a restriction because the relations (2.42) do not depend on the  $g_k$ s), we find that

$$h = \frac{2^{2N-2\sum_Q \nu_Q}}{g_1^{N-\sum_Q \nu_Q} U_0} \langle S \rangle^{N-\sum_Q \nu_Q} \prod_Q \left( b_Q + \sqrt{b_Q^2 - 4\langle S \rangle/g_1} \right)^{2\nu_Q-1}. \quad (3.33)$$

The  $\nu_Q$ s were defined in 2.2.

We can now use the operator relation (3.23), that we have derived using the rank zero vacua, to fix the function  $h$ . We obtain that  $U_0 h$  must be a function of  $\mathbf{q}$  only,

$$U_0 h(\langle S \rangle_{r=1}, \mathbf{g}, \mathbf{b}) = \frac{\mathbf{a}(\mathbf{q})}{N+1+\mathbf{a}(\mathbf{q})} \iff \mathbf{a}(\mathbf{q}) = (N+1) \frac{U_0 h}{1-U_0 h}. \quad (3.34)$$

Even though we do not really need it, let us briefly explain how the precise relation between  $\mathbf{a}$  and  $\mathbf{q}$ , equation (3.24), can be obtained. The idea is to compute the glueball superpotential. On the one hand, as reminded in 2.4, this superpotential is fixed by the anomaly equations modulo the addition of an arbitrary function  $f(S)$  that depends on the glueball field  $S$  but not on the couplings  $\mathbf{g}$ ,  $\mathbf{b}$  or  $\mathbf{q}$ . On the other hand, the glueball superpotential can be computed unambiguously from  $\langle S \rangle_{r=1}$  which is given by (3.34). It is then straightforward to check that consistency between the two results implies (3.24). A very simple way to understand why this must be valid, without performing any explicit calculation, is as follows [10]. The equation (3.34) has been obtained by implementing consistently the constraints from the  $U(1)_R$  symmetry of the theory, see the charge assignments (2.39). This symmetry also implies that the glueball superpotential must satisfy the differential equation

$$S \frac{\partial W}{\partial S} + \sum_{k \geq 0} g_k \frac{\partial W}{\partial g_k} = W. \quad (3.35)$$

As emphasized in the Section 4 of [10], this differential equation fixes the coupling-independent part  $f(S)$  in  $W(S, \mathbf{g}, \mathbf{b}, \mathbf{q})$  up to a linear term in  $S$  that corresponds to

an overall numerical factor that may multiply  $\mathbf{q}$ . In this approach, the numerical factor can be fixed by performing a single one-instanton calculation, for example in the Coulomb vacuum discussed in 3.6, and one finds again that  $h = q$ .

So we know that the relation (2.46) is valid in the rank one vacua. Using the decomposition (3.11), this is equivalent to the constraints

$$\mathcal{P}_p^{(1)}(\langle \mathbf{x} \rangle_{r=1}, \mathbf{b}, \mathbf{q}) = 0, \quad p \geq 1, \quad (3.36)$$

which is similar to (3.12), but now for the rank one vacua. To analyse the algebraic consequences of (3.36), we shall use the lemma of Section 3.3. To do this, let us first describe the classical limits of the rank one vacua. They correspond to having  $p$  of the  $x_i$ s, say  $x_1, \dots, x_p$ , to be equal to  $p$  distinct  $b_j$ s, for example  $x_i = b_i$  for  $1 \leq i \leq p$ , and to having all the other  $x_i$ s,  $i > p$ , to be equal to the same root  $w$  of the polynomial  $W'$  given in (2.1),  $x_{p+1} = \dots = x_N = w$ . The roots of  $W'$  are algebraically independent from  $\mathbf{b}$  and  $\mathbf{q}$ , and thus can be considered to be arbitrary indeterminates for our purposes. We are now going to prove a result which is the analogue, for the rank one vacua, of the proposition of Section 3.2:

**Proposition:** *Let  $A \in \mathcal{V}_n$ ,  $A \neq 0$ , such that*

$$A(x_1 = b_1, \dots, x_p = b_p, x_{p+1} = w, \dots, x_N = w, \mathbf{b}, \mathbf{q}) = 0 \quad (3.37)$$

*for all  $0 \leq p \leq N - 1$ . Then  $\deg A = n \geq 2N + 6$ .*

The proof is very similar to the one given in 3.2 after (3.17). The assumptions in the proposition imply that there exists  $k_0 \leq N - 1$ , defined to be the smallest integer such that  $A(x_1 = b_1, \dots, x_{k_0} = b_{k_0}, x_{k_0+1}, \dots, x_N, \mathbf{b}, \mathbf{q})$  vanishes. Seeing  $A(x_1 = b_1, \dots, x_{k_0-1} = b_{k_0-1}, x_{k_0}, \dots, x_N, \mathbf{b}, \mathbf{q})$  as a polynomial in  $x_{k_0}$ , using the symmetry in the  $b_k$ s for  $k \geq k_0$  and then using the symmetry in the  $x_i$ s for  $k_0 \leq i \leq N$ , we deduce that

$$A(x_1 = b_1, \dots, x_{k_0-1} = b_{k_0-1}, x_{k_0}, \dots, x_N, \mathbf{b}, \mathbf{q}) = \prod_{i=k_0}^N \prod_{k=k_0}^{2N} (x_i - b_k) B \quad (3.38)$$

for some non-zero  $B$ . Using the homogeneity of  $A$ , we get

$$\deg A \geq (2N - k_0 + 1)(N - k_0 + 1). \quad (3.39)$$

This yields

$$\deg A \geq 3(N + 3) \quad \text{if } k_0 \leq N - 2. \quad (3.40)$$

If  $k_0 = N - 1$ , (3.37) implies that  $A(x_1 = b_1, \dots, x_{N-2} = b_{N-2}, x_{N-1}, x_N)$  not only vanishes at  $x_{N-1} = b_i$  for  $i \geq N - 1$ , but also at  $x_{N-1} = x_N$ . Using in particular the

symmetry in exchanging  $x_{N-1}$  and  $x_N$ , we get

$$A(x_1 = b_1, \dots, x_{N-2} = b_{N-2}, x_{N-1}, x_N) = (x_{N-1} - x_N)^2 \prod_{k=N-1}^{2N} (x_{N-1} - b_k)(x_N - b_k)B \quad (3.41)$$

for some non-zero  $B$ , and thus

$$\deg A \geq 2(N+2) + 2 = 2N+6 \quad \text{if } k_0 = N-1. \quad (3.42)$$

Together with (3.40), this proves the proposition.

An immediate corollary is that, if  $A \in \mathcal{V}_n$  satisfies (3.37) and  $n \leq 2N+5$ , then  $A = 0$ . We can thus apply the lemma of Section 3.3, for the subset  $I$  of all the rank two vacua, to deduce from (3.36) that  $\mathcal{P}_p^{(1)}$  must vanish if  $\deg \mathcal{P}_p^{(1)} = N+p \leq 2N+5$ , or  $p \leq N+5$ .

Let us summarize what we have done. The anomaly equations imply that the relations

$$\langle u_{N+p} \rangle_{r=1} = \mathcal{P}_p^{(0)}(\langle u_1 \rangle_{r=1}, \dots, \langle u_N \rangle_{r=1}, \mathbf{b}, \mathbf{q}) \quad (3.43)$$

are valid for all  $p \geq 1$  in the rank one vacua. This implies that the relations

$$u_{N+p} = \mathcal{P}_p^{(0)}(u_1, \dots, u_N, \mathbf{b}, \mathbf{q}) \quad (3.44)$$

are valid as *operator relations* for

$$1 \leq p \leq N+5. \quad (3.45)$$

Equivalently, (2.46) must be valid as an operator relation up to terms of order  $z^{-N-6}$ ,

$$F(z) + \frac{qU(z)}{F(z)} = H(z) + \mathcal{O}(1/z^{N+6}) \quad (3.46)$$

for some degree  $N$  polynomial  $H$ .

### 3.5 Using the rank two vacua

Operator relations are valid in all the vacua, and thus (3.46) yields  $N+5$  non-trivial constraints on the solutions (2.38). This is enough to fix completely  $\langle R(z) \rangle_r$  as long as the number of parameters is smaller than the number of constraints, i.e. when  $2r-1 \leq N+5$ . In particular, we know all the rank two solutions.

Concretely, taking the derivative of (3.46) with respect to  $z$ , we find that (2.50) must be valid in the rank two vacua up to terms of order  $1/z^{2N+7}$ . Comparing with (2.38), we thus obtain

$$\begin{aligned} \frac{C}{y_2} + \frac{1}{2} \frac{U'}{U} - \frac{1}{2y_2} \sum_Q \frac{(1 - 2\nu_Q)y_2(z = b_Q)}{z - b_Q} = \\ \frac{1}{2} \frac{U'}{U} + \left( H_2' - \frac{U' H_2}{2U} \right) \frac{1}{\sqrt{H_2^2 - 4qU}} + \mathcal{O}(1/z^{2N+7}), \end{aligned} \quad (3.47)$$

for some constant  $C$  and where we have defined  $H_2 = \langle H \rangle_2 = (1 + \mathbf{q})z^N + \dots$ . By expanding at large  $z$ , (3.47) yields  $2N + 5$  non-trivial constraints, which is more than enough to determine the  $N + 3$  free parameters in  $y_2$ ,  $C$  and  $H_2$  (actually, we only need (3.47) up to terms of order  $1/z^{N+5}$ ). We have checked explicitly that the solution is indeed uniquely fixed, in the particular case of the rank two vacuum corresponding to the classical limit

$$\langle R(z) \rangle_{2, \text{cl}} = \sum_{i=1}^{N-2} \frac{1}{z - b_i} + \frac{1}{z - w_1} + \frac{1}{z - w_2}. \quad (3.48)$$

Note that we shall use only this vacuum in the following. Performing the check is completely straightforward, but quite tedious. The idea is to study  $\langle R(z) \rangle_2$  in a small  $q$  expansion around the classical solution (3.48), using (3.47). A recursive argument shows that the expansion parameter is  $q$  (not a fractional power of  $q$ ), as expected in this weakly coupled vacuum with unbroken gauge group  $U(1)^2$ , and that the coefficients in the small  $q$  expansion are uniquely fixed to all orders by the constraints (3.47). We have not tried, however, to work out directly the explicit form of the solution to all orders from (3.47). Actually, this is not necessary. A solution to (3.47) is known, and corresponds to imposing the factorization condition (2.52) at  $r = 2$ . The uniqueness of the solution then ensures that the solution obtained in the small  $q$  expansion must correspond to this factorization condition. But the factorization condition is equivalent to the validity of the quantization conditions (2.43) and (2.44), or to the fact that (3.47) is actually true to *all* orders, or also to the relation

$$\langle F(z) \rangle_2 + \frac{qU(z)}{\langle F(z) \rangle_2} = \langle H(z) \rangle_2 \quad (3.49)$$

in the rank two vacua. Strictly speaking, we have proven (3.49) only in the rank two vacua with classical limits (3.48), but this is enough for our purposes.

Using (3.11), the equation (3.49) is equivalent to

$$\mathcal{P}_p^{(1)}(\langle \mathbf{x} \rangle_2, \mathbf{b}, \mathbf{q}) = 0, \quad p \geq 1. \quad (3.50)$$

This is the rank two version of (3.12) and (3.36). We can thus proceed along the lines of Sections 3.2 and 3.4. We shall use the:

**Proposition:** *Let  $A \in \mathcal{V}_n$ ,  $A \neq 0$ , such that*

$$A(x_1 = b_1, \dots, x_{N-2} = b_{N-2}, x_{N-1}, x_N, \mathbf{b}, \mathbf{q}) = 0. \quad (3.51)$$

*Then  $\deg A = n \geq 3N + 9$ .*

The proof goes as after (3.17) or (3.37). Using the symmetry properties of the variables in  $A \in \mathcal{V}_n$ , the constraints (3.51) imply that there exists  $k_0 \leq N - 2$  such that

$$A(x_1 = b_1, \dots, x_{k_0-1} = b_{k_0-1}, x_{k_0}, \dots, x_N, \mathbf{b}, \mathbf{q}) = \prod_{i=k_0}^N \prod_{k=k_0}^{2N} (x_i - b_k) B \quad (3.52)$$

for some non-zero  $B$ , and thus

$$\deg A \geq (2N - k_0 + 1)(N - k_0 + 1) \geq 3(N + 3) \quad (3.53)$$

as we wished to show.

The condition (3.51) corresponds to the classical vacua (3.48) (note that the roots  $w_1$  and  $w_2$  of  $W'$  are algebraically independent from the  $b_Q$ s and  $\mathbf{q}$ , and thus play the rôle of independent variables). We can thus use the lemma of Section 3.3 to conclude that (3.50) implies that  $\mathcal{P}_p^{(1)} = 0$  for all  $p \leq 2N + 8$ , or equivalently that

$$F(z) + \frac{qU(z)}{F(z)} = H(z) + \mathcal{O}(1/z^{2N+9}) \quad (3.54)$$

must be valid as an operator relation. The  $2N + 8$  non-trivial operator relations that follow from (3.54) are more than enough to fix unambiguously the free parameters in (2.38), in *all* the possible cases. Indeed, the maximal rank is  $N$ , and the maximum number of parameters that can appear in (2.38) is thus  $2N - 1$ .

### 3.6 Using the rank $N$ vacuum

The proof of our main theorem is now at hand. Let us analyse the rank  $N$  Coulomb vacuum. Classically, this vacuum corresponds to

$$\langle R(z) \rangle_{N, \text{cl}} = \sum_{i=1}^N \frac{1}{z - w_i}, \quad (3.55)$$

where the  $w_i$ s are the root of  $W'$ , see (2.1). Quantum mechanically, the solution is uniquely fixed by (3.54) (the validity of this equation is actually needed only up to

terms of order  $1/z^{2N}$ ). This is shown as in the rank two case, using the analogue of (3.47). The unique solution must correspond to the known one, which is characterized by the condition (2.52) at  $r = N$  (in this case  $\psi_0$  is just a constant and  $y_N^2 = W'^2 - 4\Delta_{N-1}$ ). We deduce that the relation

$$\langle F(z) \rangle_N + \frac{qU(z)}{\langle F(z) \rangle_N} = \langle H(z) \rangle_N \quad (3.56)$$

is valid in the rank  $N$  vacuum, or equivalently that

$$\mathcal{P}_p^{(1)}(\langle \mathbf{x} \rangle_N, \mathbf{b}, \mathbf{q}) = 0, \quad p \geq 1. \quad (3.57)$$

However, in the Coulomb vacuum, *the  $\langle x_i \rangle$  are algebraically independent* from the  $b_{\mathbf{q}s}$  and  $\mathbf{q}$  in the classical limit (3.55). Combining this fact together with (3.57) and the lemma in Section 3.3, we get

$$\mathcal{P}_p^{(1)}(\mathbf{x}, \mathbf{b}, \mathbf{q}) = 0. \quad (3.58)$$

This completes the proof of the chiral ring consistency theorem.

## 4 Conclusions

The chiral ring consistency theorem sheds considerable light on the inner workings of the gauge theory/matrix model correspondence. In the matrix model, the planar limit must be taken and thus the variables that enter the loop equations are all independent. As a consequence, the most general solution is parametrized by arbitrary filling fractions. In the gauge theory, the number of colours  $N$  is finite, relations like (2.42) must exist, and there is only a finite number of independent variables. Consistency between the matrix model loop equations, that are mapped onto the gauge theory generalized Konishi anomaly equations, and the gauge theory identities (2.42), is then possible only for some particular values of the filling fractions. These correspond to the expectation values of the gauge theory glueball superfields and encode a very rich non-perturbative dynamics.

As explained in the introduction, our results also illustrate a deep consistency property of the open/closed string duality. The closed string results can be written in the open string language if and only if the closed string superpotential is extremized. Algebraic identities in the open string picture and closed string equations of motion are exchanged in the duality.

The general line of thinking used in the present paper was already used in [8]. The argument in this earlier work was that the relations (2.42) can be determined by



looking at the weakly coupled Coulomb vacuum, because of the algebraic independence of the variables in this case. At least in principle, everything can be computed in this vacuum by performing explicit instanton calculation. Since the relations (2.42) are operator equations, they must then be valid in all the other vacua of the theory, including the strongly coupled vacua where the semi-classical approximation does not apply. Since they are equivalent to the quantization conditions (2.43) and (2.44), the latter must also be valid in all the vacua of the theory. The main contribution of the present paper, with respect to [8], is to show that the explicit calculations in the Coulomb vacuum, which require considerable technology, are not necessary if one starts from the non-perturbative anomaly theorem. Everything is then fixed by the internal algebraic consistency of the chiral ring.

We believe that the general philosophy of the present work applies to *any*  $\mathcal{N} = 1$  supersymmetric gauge theory, including in the cases where there is a moduli space of vacua. By combining the quantum version of the classical equations of motion written in terms of the gauge invariant observables, which are the generalized Konishi anomaly equations, with the full set of identities that follow from the definition of these variables in terms of fields transforming non-trivially under the gauge group, one should be able to determine unambiguously all the quantum vacua and associated chiral operators expectation values.

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